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Generalized viscous shock layer equations with slip conditions on a surface in a flow field and on a head shock☆

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ABSTRACT

The plane and axisymmetric problems of super- and hypersonic flow of a homogeneous viscous heat-conducting perfect gas over a blunt body are considered. Generalized viscous shock layer equations that take into account all the second-order effects of boundary-layer theory, i.e., the terms $O(\mathrm{Re}^{-1/2})$, are derived from the Navier–Stokes equations by the asymptotic method, and all the out-of-order third-order terms $O(\mathrm{Re}^{-1})$ and higher-order terms are also retained, except terms with second derivations in the marching coordinate (Re is Reynolds number, determined from the free-stream density and velocity the linear dimension, which is equal to the nose radius of the blunt Body, and the free-stream shear viscosity at the stagnation temperature). Thus, only the presence of terms with second derivatives in the marching coordinate, which specify the elliptical properties of the complete system of Navier–Stokes equations, distinguish it from the generalized viscous shock layer equations, which do not contain these terms. Slip and a temperature jump conditions on a body surface are presented with the same degree of accuracy, and generalized Rankine–Hugoniot conditions on a head shock, which take into account the effects of the viscosity and heat conduction, including their influence on the determination of the pressure, are derived. The incorrect and unfounded approximations used in preceding studies and the efficiency of iterative marching techniques for solving the generalized viscous shock layer equations, as well as the ability of the latter to provide a correct solution for the drag and heat-transfer coefficients in the transitional flow regime if the solution is constructed taking the slip and temperature jump on a surface and on a head shock into account, are noted.

Many versions of the Navier–Stokes equations for solving various external and internal problems in the thermodynamics of a viscous heat-conducting gas, that were simplified by the asymptotic and other approximation methods using the properties of the specific problem, were proposed in the nineteen seventies and eighties.[1–8] Calculations of steady flows based on the complete Navier–Stokes equations by the relaxation method are fairly time-consuming, especially for flows of multicomponent, chemically reacting gases. For high Reynolds numbers and an integration region of great length (long bodies), the requirements for computer resources increase considerably. For example, calculations of a simple gas-dynamical steady flow by the relaxations method using the most efficient monotone implicit difference schemes require the several hundred time iterations.[4,5] At the same time, for separation-free viscous flows, the need to use the complete Navier–Stokes equations arises only at low Reynolds numbers. However, in this case, too, even with the appearance of flow separations and especially at high Reynolds numbers, the description of such a flow with sufficient accuracy is possible within simpler mathematical models that require significantly less computational resources in many cases of practical importance. Therefore, along with the use of the complete system of Navier–Stokes equations, simplified Navier–Stokes equations that allow the use of efficient evolution–marching-through-space methods[9–12] for their numerical solution are widely used for calculating flows in nozzles and channels and in problems of external flow over a body.

For external flows there are numerous such models.[13] Simplified equations are obtained from the Navier–Stokes equations mainly by discarding terms with the viscosity and thermal conductivity of a different order in the small parameter $\mathrm{Re}^{-1/2}$. However, the common distinguishing feature of the simplified equations is the absence of second derivatives of the components of the velocity vector and the temperature (enthalpy) along the marching coordinate aligned in the predominant direction of the gas flow in the terms with the viscosity and thermal conductivity. There is consequently the possibility of obtaining a numerical solution of steady-state problems by

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efficient iterative marching methods, which is especially important when calculating flows, taking into account the actual physicochemical processes that occur in a shock layer and on a surfac.[6,14,15]

The most widely known approaches for problems of external flow over a body are the model of a thin viscous shock layer,[1] the model of a full viscous shock layer,[16,17] The parabolized Navier–Stokes equations,[3] the parabolic-hyperbolic model of a viscous shock layer[8,12], etc.

In this paper new, more meaningful simplified Navier–Stokes equations are derived. When $\mathrm{Re}\to\infty$, the $O(1)$ and $O(\mathrm{Re}^{-1/2})$ terms and the $O(\mathrm{Re}^{-1})$ terms that do not increase the order of the derivatives along the marching coordinate to two are retained, and the appearance of elliptical properties in the simplified system of equations due to consideration the the viscosity and heat condition effects, which are inherent in the complete Navier–Stokes equations, is thereby eliminated. However, in the simplified Navier–Stokes equations elliptical properties show up in subsonic flow regions, making it necessary to employ some iterative methods of global iterations.[9,10] The most efficient among them is the method described in Refs 11 and 12, which is based on special splitting of the marching component of the pressure gradient into hyperbolic and elliptical components on the difference level; the number of required iterations for the elliptical component is reduced to one or two, which is also the total number of global iterations. This method is also applicable for solving the generalized full viscous shock layer equations obtained in this paper. In addition, recent results showed[18,19] that consideration of the slip and temperature jump on a surface and on a head shock greatly extend the applicability of simplified Navier–Stokes equations for solving problems of high-altitude aerothermodynamics into the region of the transitional flow regime. Therefore, exact generalized Rankine–Hugoniot conditions, which have been obtained in the literature and have been used either in approximate formulations[2,5,20] or with errors,[17] are also derived below.

1. Navier–Stokes equations in a natural system of coordinates connected to a body surface

Consider the two-dimensional problem of the steady laminar flow of a viscous heat-conducting perfect gas over a blunt axisymmetric or blunt plane body, ignoring the actual physicochemical processes that occur at high velocities in a shock wave and on a surface in a flow field, whose consideration does not alter the mathematical nature of the problem, but significantly complicates its numerical implementation. Bearing in mind the goal of extending the model cited in the title of this paper to low Reynolds numbers, we will also formulate boundary conditions with slip and a temperature jump on a surface and on a head shock (the generalized Rankine–Hugoniot conditions at low Reynolds numbers).

In problems concerned with a flow over blunt bodies, it is convenient to use an orthogonal curvilinear system of coordinates connected to the body surface.[7] Assuming that the contour of a plane or axisymmetric body is fairly smooth (at each point of the contour, only one definite tangent plane or normal to the contour can be constructed with a possible discontinuity of the curvature of the contour), we will consider the translational steady supersonic gas flow over it with a velocity V_∞ along the axis of symmetry Oz of the body (Fig. 1), in an orthogonal curvilinear system of coordinates connected to its surface. In this system of coordinates, the position of the point P in the flow near the body is specified by its distance $yR_0 = PN$ from the body surface at the point N along the outward normal to the contour and by the length of the arc $xR_0 = ON$ along the contour from its vertex O to the base N of the normal. Here $R_0 = R(0)$ is the radius of curvature of the body at its vertex, and x and y are the dimensionless coordinates along the contour and along the normal to the contour. The third coordinate φ will specify the position of the meridian plane passing through the Oz axis.

We will write the fundamental system of equations of the gas motion in dimensionless form, after introducing the dimensionless parameters of the flow: $\rho_\infty\rho$ is the mass density of the gas, $\upsilon_x = V_\infty u$ and $\upsilon_y = V_\infty \upsilon$ are the physical components of the velocity vector along the x and y axes, $\rho_\infty V_\infty^2 p$ is the pressure, $\mu\,\mu_0$ is the coefficient of shear viscosity, and $\zeta\,\mu_0$ is the coefficient of bulk viscosity. Here ρ, u, υ, p, μ and ζ are dimensionless flow parameters, viz., the density, the velocity components, the pressure, the coefficient of shear viscosity and the coefficient of bulk viscosity, and μ_0 is the characteristic value of the viscosity coefficient, for example, the viscosity coefficient at the adiabatic stagnation temperature of a perfect gas T_0 or another appropriate characteristic value of it. All the quantities with the dimension of length will be related to R_0. In the system of coordinates chosen the Navier–Stokes equations written in the dimensionless parameters just listed for the plane ($\nu = 0$) and axisymmetric ($\nu = 1$) problems will be written in the following form:[7]

the continuity equation

$$\frac{\partial}{\partial x}\left(r^\nu \rho u\right) + \frac{\partial}{\partial y}\left(H_1 r^\nu \rho \upsilon\right) = 0 \tag{1.1}$$

the momentum equations projected onto the x and y axes

$$\rho\left(\frac{u}{H_1}\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + \frac{u\upsilon}{RH_1}\right) = -\frac{1}{H_1}\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}(\nabla\cdot\hat{\tau})_x \tag{1.2}$$

$$\rho\left(\frac{u}{H_1}\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial v}{\partial y} - \frac{u^2}{RH_1}\right) = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}(\nabla\cdot\hat{\tau})_y \tag{1.3}$$

the energy equation, written in terms of the total dimensionless enthalpy

$$\rho\left(\frac{u}{H_1}\frac{\partial H}{\partial x} + \upsilon\frac{\partial H}{\partial y}\right) + \frac{1}{\mathrm{Re}}(\nabla\cdot\mathbf{J}) = 0;\quad H = h + \frac{u^2+\upsilon^2}{2} \tag{1.4}$$

where $V_\infty^2 H$ is the total (dimensional) enthalpy, $V_\infty^2 h$ is the thermodynamic (dimensional) enthalpy, and H and h are dimensionless quantities,

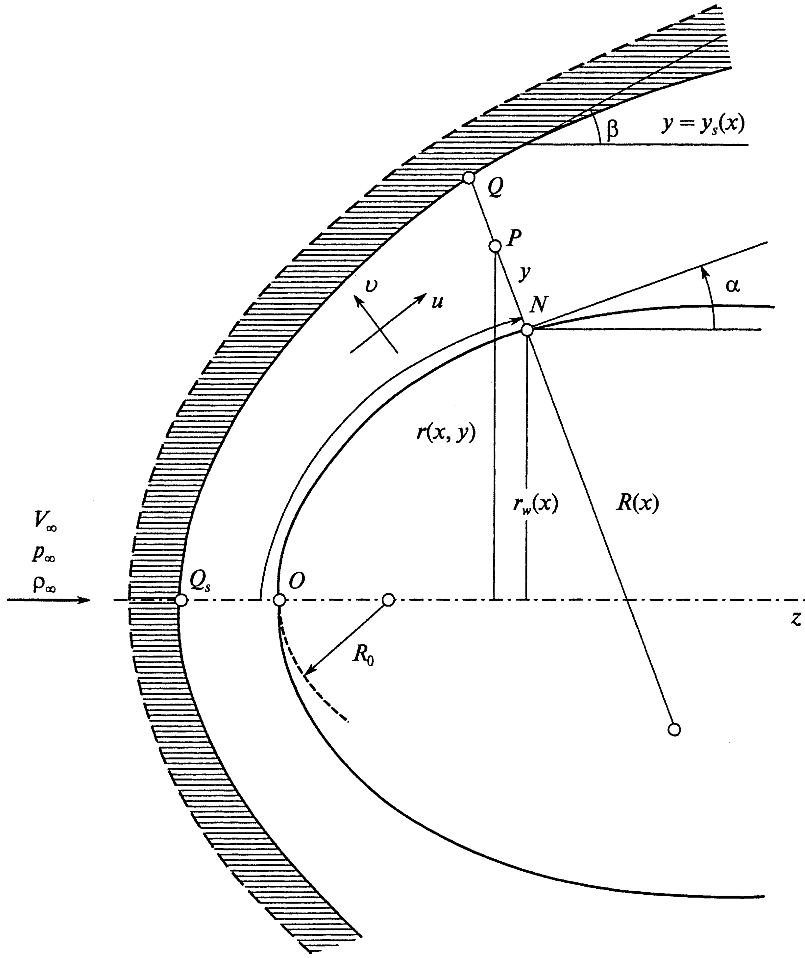

Fig. 1.

the state equation for a perfect gas

$$p = \rho R_A T / V_\infty^2 \tag{1.5}$$

where R_A is the specific absolute gas constant (per unit mass), mR_A is the absolute gas constant, m is the molecular weight of the gas, and T is the absolute temperature.

Dimensionless equations (1.2)–(1.4) contain the Reynolds number $\mathrm{Re} = \rho_\infty V_\infty R_0/\mu_0$, which is determined from the free-stream parameters and the characteristic coefficient of shear viscosity μ_0.

By virtue of the symmetry of the problem, we have

$$\upsilon_\varphi = 0, \quad \frac{\partial}{\partial\varphi} \equiv 0, \quad H_1 = 1 + \frac{y}{R(x)}, \quad H_2 = H_y = 1, \quad H_3 = H_\varphi = r^\nu = [r_w(x) + y\cos\alpha(x)]^\nu$$

(H_1, H_2 and H_3 are the Lamé coefficients of the metric used). Here $R(x)$ is the dimensionless radius of curvature of the body contour, $r_w(x)$ is the distance from a point on the body contour to the axis of symmetry Oz, and $\alpha(x)$ is the angle between a tangent to the body contour and the axis of symmetry Oz (Fig. 1). The functions that characterize the geometry of the body $r_w(x)$, $\alpha(x)$, $R(x)$, and also the distance $r(x, y)$ of the point I in the flow field from the axis of symmetry Oz for a convex body with possible curvature discontinuities on its contour are related by the evident geometrical relations

$$r_w(x) = \int_0^x \sin\alpha(t)dt = \int_{\alpha(x)}^{\pi/2} R(t)\sin t dt, \quad \frac{dr_w}{dx} = \sin\alpha(x), \quad \frac{dr_w}{d\alpha} = -R(x)\sin\alpha(x)$$
$$-\frac{d\alpha}{dx} = \frac{1}{R(x)}, \quad \frac{\partial r}{\partial x} = H_1\sin\alpha(x), \quad \frac{\partial r}{\partial y} = \cos\alpha(x) \tag{1.6}$$

Henceforth the argument x will be omitted in these functions.

The physical components of the divergence of the viscous stress tensor $\hat{\tau}$ along the x and y axes have the form

$$(\nabla\cdot\hat{\boldsymbol{\tau}})_x = \frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu \tau_{xx}\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu \tau_{xy}\right)\right]+\frac{\tau_{xy}}{RH_1}-\frac{\nu\sin\alpha}{r}\tau_{\varphi\varphi}$$
$$=\frac{1}{H_1}\frac{\partial}{\partial x}\tau_{xx}+\frac{1}{H_1^2 r^\nu}\frac{\partial}{\partial y}\left(H_1^2 r^\nu \tau_{xy}\right)+\frac{\nu\sin\alpha}{r}\left(\tau_{xx}-\tau_{\varphi\varphi}\right)$$
$$(\nabla\cdot\hat{\boldsymbol{\tau}})_y = \frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu \tau_{xy}\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu \tau_{yy}\right)\right]-\frac{\tau_{xx}}{RH_1}-\frac{\nu\cos\alpha}{r}\tau_{\varphi\varphi}$$
$$=\frac{1}{H_1}\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\nu\sin\alpha}{r}\tau_{xy}+\frac{1}{RH_1}\left(\tau_{yy}-\tau_{xx}\right)+\frac{\nu\cos\alpha}{r}\left(\tau_{yy}-\tau_{\varphi\varphi}\right) \tag{1.7}$$

The physical dimensionless components (dimensional components relative to $\mu_0 V_\infty/R_0$) of the viscous stress tensor τ_{ij} $(i, j = x, y, \varphi)$ in the selected system of coordinates have the following form (the generalized Newton's law)

$$\tau_{xx}=\left(\zeta-\frac{2}{3}\mu\right)\nabla\cdot\mathbf{v}+2\mu e_{xx},\quad \tau_{yy}=\left(\zeta-\frac{2}{3}\mu\right)\nabla\cdot\mathbf{v}+2\mu e_{yy},$$
$$\tau_{\varphi\varphi}=\left(\zeta-\frac{2}{3}\mu\right)\nabla\cdot\mathbf{v}+2\mu e_{\varphi\varphi}$$
$$\tau_{xy}=2\mu e_{xy},\quad \tau_{y\varphi}=0,\quad \tau_{\varphi x}=0 \tag{1.8}$$

The quantities e_{ij} $(i, j = x, y, \varphi)$, which are the physical dimensionless components (dimensional components relative to V_∞/R_0) of the rate of deformation tensor, have the form

$$e_{xx}=\frac{1}{H_1}\frac{\partial u}{\partial x}+\frac{\upsilon}{RH_1},\quad e_{yy}=\frac{\partial\upsilon}{\partial y},\quad e_{\varphi\varphi}=\frac{\nu}{r}(u\sin\alpha+\upsilon\cos\alpha)$$
$$2e_{xy}=\frac{\partial u}{\partial y}-\frac{u}{RH_1}+\frac{1}{H_1}\frac{\partial\upsilon}{\partial x},\quad e_{y\varphi}=0,\quad e_{\varphi x}=0$$
$$\nabla\cdot\mathbf{v}=\frac{1}{H_1 r^\nu}\left[\frac{\partial}{\partial x}\left(r^\nu u\right)+\frac{\partial}{\partial y}\left(H_1 r^\nu\upsilon\right)\right]=e_{xx}+e_{yy}+e_{\varphi\varphi}$$
$$\nabla\cdot\mathbf{J}=\frac{1}{H_1 r^\nu}\frac{\partial}{\partial y}\left(H_1 r^\nu J_y\right)+\frac{\nu\sin\alpha}{r}J_x+\frac{1}{H_1}\frac{\partial}{\partial x}J_x \tag{1.9}$$

The dimensionless total energy flux density vector $\mathbf{J}$ ($\mathbf{J}'$ in dimensional form) in the energy equation (1.4) written in terms of the dimensionless (relative to V_∞^2) enthalpies h and H has the following expression

$$\mathbf{J}=\frac{R_0}{\mu_0 V_\infty^2}\mathbf{J}'=-\frac{\mu c_p'}{\sigma V_\infty^2}\nabla T+\hat{\boldsymbol{\tau}}\cdot\mathbf{v}=-\frac{\mu}{\sigma}\nabla h+\hat{\boldsymbol{\tau}}\cdot\mathbf{v}=-\frac{\mu}{\sigma}\left[\nabla H+\frac{\sigma}{\mu}(\hat{\boldsymbol{\tau}}\cdot\mathbf{v})-\nabla\frac{u^2+\upsilon^2}{2}\right] \tag{1.10}$$

where

$$\hat{\boldsymbol{\tau}}\cdot\mathbf{v}=(\tau_{xx}u+\tau_{xy}\upsilon)\mathbf{e}_1+(\tau_{xy}u+\tau_{yy}\nu)\mathbf{e}_2,\quad \sigma=\mu\mu_0 c_p'/\lambda'$$

λ' is the dimensional thermal conductivity, and c_p' is the dimensional specific heat (per unit mass) of the gas at constant pressure.

The gradients of T, h and H have the form

$$\nabla f=\frac{1}{H_1}\frac{\partial f}{\partial x}\mathbf{e}_1+\frac{\partial f}{\partial y}\mathbf{e}_2,\quad f=\{T,h,H\}$$

When the temperature dependences of the coefficients ζ, μ, λ' and c_p' are assigned, the system of Navier–Stokes equations (1.1)–(1.5), taking into account transport relations (1.8) and (1.10) takes the form of a closed system of five equations for specifying the five functions u, υ, ρ, p and H (or T). Since the construction of gas-dynamic models of super- and hypersonic flow over blunt bodies in different ranges of Reynolds (Knudsen) numbers is considered here, we will ignore the physicochemical processes in the flow near the body, which unavoidably accompany problems of flow over bodies at high supersonic velocities. Methods for taking them into account have been thoroughly developed, as they do not cause any fundamental difficulties,[6,15,16] and they will not alter the fundamental conclusions regarding the establishment of gas-dynamic models for various limit values of the fundamental governing parameters of the problem, viz., the free-stream Mach and Reynolds numbers, the temperature factor, etc. Of course, consideration of the physicochemical processes will result in changes in the quantitative results; however, the gas-dynamic model itself will not be altered, since its form and properties depend on the method used to simplify only the fundamental equations (1.2)–(1.4) and the equations of convective diffusion, which are added to these equations if chemical reactions and ionization reactions are taken into account.[15]

An important conclusion follows at once from the second form of representing viscosity terms (1.7). If the first terms in expressions (1.7), which specifically provide the second derivatives in longitudinal coordinate x (the elliptical terms) in the Navier–Stokes equations, and the second term in the expression for $(\nabla\cdot\hat{\tau})_y$, which drops out in the viscous shock layer model, are ignored, the viscosity terms will not contain the bulk viscosity coefficient. In fact, the normal stresses on areas on the coordinate surfaces x = const, y = const and φ = const and the quantities τ_{xx}, τ_{yy} and $\tau_{\varphi\varphi}$, which contain the bulk viscosity coefficient ζ, appear in the form of differences, and, therefore, the terms containing this coefficient (see (1.8)) cancel out. Thus, the terms with the bulk viscosity coefficient are attributed to effects of a third approximation that are of the order of Re^{-1}.

2. Boundary conditions

As an object (a spacecraft, meteoroid, etc.) moves through the atmosphere of the Earth or other planets, it successively encounters different regimes of flow over it: free-molecule, transitional and continuum flow. Here we will confine ourselves to considering the last two flow regimes, for which we can use continuum models. In free-molecule flow, Boltzmann's equation has an exact solution in the form of the equilibrium Maxwell distribution function, and determination of the drag and heat transfer then reduces to quadratures under known boundary conditions. This is, perhaps, the solely exact solution of the problem of gas mechanics.

At fairly high Reynolds numbers, under the condition of a specified injection or suction normal to the surface of a gas, whose physical properties are identical to the free-stream gas properties, within the statement of the problem given, which does not take into account chemical processes in the gas or on the body field, the boundary conditions for the velocity will be

$$u(x,0)=0,\quad (\rho_\infty \rho y_2)_w = Q(x) \tag{2.1}$$

where $Q(x)$ is the specified gas mass flow rate distributed over the surface per unit area.

The temperature

$$T(x,0)=T_w(x) \tag{2.2}$$

or the energy balance condition

$$\left[\frac{\mu}{\sigma}\left(\frac{\partial H}{\partial y}-\upsilon\frac{\partial \upsilon}{\partial y}\right)+\upsilon\tau_{yy}\right]_{y=0}=\frac{R_0\varepsilon}{\mu_0 V_\infty^2}\sigma_B T_w^4(x) \tag{2.3}$$

is specifed on the body surface to determine the equilibrium radiant temperature of the surface. Here ε is the emissivity of the body surface, σ_B is the Stefan–Boltzmann constant, and $T_W(x)$ is the assigned temperature (condition (2.2)) or sought temperature (condition (2.3)) of the body surface. Condition (2.3) was written under the assumption that the heat flux into the body is negligibly small compared with the radiation from the surface. This condition holds with good accuracy when there is heat exchange between the gas and the surface for bodies moving at high velocities in the atmosphere. In addition, in formulating boundary conditions (2.1)–(2.3) it was assumed that there are no jumps in the tangential component of the velocity or the temperature of the gas on the surface, that is justified for sufficiently high Reynolds numbers.

We will now write out a heat-balance relation that extends condition (2.3) to the case of ablation of material in gaseous form without the formation of a liquid film (or neglecting its influence), i.e., for a sublimation process. It will have the form[21]

$$\left[\frac{\mu}{\sigma}\left(\frac{\partial H}{\partial y}-\upsilon\frac{\partial \upsilon}{\partial y}\right)+\upsilon\tau_{yy}\right]_{w}-\frac{R_0\varepsilon\sigma_B}{\mu_0 V_\infty^2}T_w^4(x)=\frac{R_0}{\mu_0 V_\infty^2}(\rho_\infty\rho\upsilon_y)_w H_{\mathrm{eff}};\ H_{\mathrm{eff}}=\Delta+h_w-h_{-\infty} \tag{2.4}$$

Here Δ is the specific heat of the phase transition of the vaporized material of the surface, and $h_W - h_{-\infty}$ is the enthalpy drop of the body material from the temperature of the phase transition to the initial temperature of the body. However, condition (2.4) does not close the problem since the unknown temperature T_W appears in it. It must be related to the partial vapour pressure, the vapour pressure curve[21] must be used, and the convective diffusion equation for the vapour must be added. The boundary conditions written above specify the solution in the region of the thermally stressed part of the entry trajectory of a spacecraft or for fairly large meteoroids, from which maximum ablation occurs under the conditions of continuum flow regime. As the transitional flow regime is approached, i.e., as the Reynolds number decreases (the Knudsen number increases), flow regions that are far from equilibrium with respect to the translational degrees of freedom begin to appear. This refers, first and foremost, to the appearance of a non-equilibrium Knudsen layer directly near the surface and a fairly thick non-equilibrium structure of the head shock. The flows in these regions are described by kinetic equations, whose solution should be combined with the solution of the continuum equations (with the solution of the Navier–Stokes equations or with asymptotically simplified versions of them). Solving any kinetic equation in the Knudsen layer and combining it with the solution of the Navier–Stokes equations give the slip and a temperature jump conditions on the outer boundary of the Knudsen layer, which can be extrapolated to the wall and whose simplest version has the following form[22,23] (the boundary conditions of the third kind)

$$u(x,0)=\frac{2-\theta}{\theta}\sqrt{\frac{\pi}{2}}\frac{\mu}{\mathrm{Re}\sqrt{p\rho}}\frac{\partial u}{\partial y},\quad T(x,0)=T_w+\frac{2-\alpha}{\alpha\sigma}\frac{2\gamma}{\gamma+1}\sqrt{\frac{\pi}{2}}\frac{\mu}{\mathrm{Re}\sqrt{p\rho}}\frac{\partial T}{\partial y} \tag{2.5}$$

where θ is the diffuse reflection coefficient and α is the energy accommodation coefficient.

When slip and a temperature jump appear, the dimensionless tangential (frictional) stresses on the surface (1.8) and (1.9) and the dimensionless heat flux (1.10) should be found from the expressions (taking into account the gas injection from the surface, i.e.,

$\upsilon(x,0) \neq 0)$

$$\tau(x,0) = 2\mu_w e_{xy}(x,0) = \mu_w\left(\frac{\partial u}{\partial y} - \frac{u}{R} + \frac{\partial \upsilon}{\partial x}\right)_w$$

$$\frac{R_0}{\mu_0 V_\infty^2} J_w = \frac{\mu_w}{\sigma_w}\left[\frac{\partial H}{\partial y} - u\frac{\partial u}{\partial y} - \upsilon\frac{\partial \upsilon}{\partial y}\right]_w + (\tau_{xy}u + \tau_{yy}\upsilon)_w$$

and the right-hand sides of these expressions should replace the derivatives $\partial u/\partial y$ and $\partial T/\partial y$ in conditions (2.5), respectively.

When a supersonic flow over a body is considered the free-stream velocity vector and the functions p, ρ and H (or T), that have been reconciled with the equation of state (1.5), should be specified for the Navier–Stokes equations:

$$\begin{aligned}
&\upsilon_x[x, y_s(x)] = \upsilon_{x\infty} = V_\infty \cos\alpha, \quad u[x, y_s(x)] = \cos\alpha \\
&\upsilon_y[x, y_s(x)] = \upsilon_{y\infty} = V_\infty \sin\alpha, \quad \upsilon[x, y_s(x)] = \sin\alpha \\
&\rho[x, y_s(x)] = 1, \quad p[x, y_s(x)] = 1/(\gamma M_\infty^2), \quad \gamma = c_p/c_\upsilon \\
&H'[x, y_s(x)] = H_\infty = h_\infty + (\upsilon_{x\infty}^2 + \upsilon_{y\infty}^2)/2 \quad (\text{or } T[x, y_s(x)] = T_\infty)
\end{aligned} \tag{2.6}$$

Here $y = y_s(x)$ is the specified hypothetical boundary, which is located fairly far in front of the body and on which the parameters of the unperturbed free stream are specified, c_υ is the specific heat at constant volume, which, like c_p, will be regarded as a constant equal to $c_\upsilon = c_{\upsilon\infty}$, and γ is the adiabatic exponent (in the free stream $\gamma = \gamma_\infty$), the prime indicates that the quantity is dimensional.

When a supersonic flow over a body is considered at fairly high Reynolds numbers (Re), for which the thickness (which is proportional to Re^{-1}) and thus the structure of the head shock can be neglected, it is convenient to replace the free-stream boundary conditions by corresponding conditions on the shock in order not to numerically pass through a thin Navier–Stokes shock wave structure with large gradients. When $\mathrm{Re} \to \infty$, these conditions will be the ordinary Rankine–Hugoniot conditions on a strong discontinuity, which will give a one-parameter family of solutions at specified free-stream parameters for $u[x, y_s(x)]$, $\upsilon[x, y_s(x)]$, $p[x, y_s(x)]$, $\rho[x, y_s(x)]$ and $H[x, y_s(x)]$ (or $T[x, y_s(x)]$) immediately behind the shock, which depend on the angle of inclination (a parameter) of the shock $\beta(x)$. The angle $\beta(x)$ and the shock detachment $y_s(x)$ are related by the trivial geometric relation (Fig. 1)

$$dy_s/dx = H_{1s}\,\mathrm{tg}\beta_s, \quad H_{1s} = 1 + y_s(x)/R(x), \quad \beta_s = \beta - \alpha \tag{2.7}$$

where $\beta_s(x)$ is the angle of inclination of a tangent to the shock wave relative to a tangent to the x coordinate line.

Equation (2.7) relates two independent quantities: $\beta_s(x)$ and $y_s(x)$. Therefore, in formulating the problem of supersonic flow over a body within the complete Navier–Stokes equations (1.1)–(1.5), which is of the seventh order in the coordinate y, with four boundary conditions for u, υ, H (or T) on the shock (see relations (2.6)) and three conditions on the body for u, υ and T, one condition for the shock wave detachment $y_s(x)$ to be determined is locking.

Nevertheless, in such formulations of the problem (the solution of the complete Navier–Stokes equations with a boundary on the shock instead of specifying free-stream boundary conditions fairly far from the body), the additional condition (which does not follow from the mechanical formulation of the problem) on the wall $\partial p/\partial y(x,0) = 0$ is often imposed in the literature, or such formulations of the problem are self-consistent at a difference level, as has been done by different researchers in various ways[5] with resultant non-uniqueness of the solution. There is still an alternative for avoiding this artificial non-uniqueness: either solve the problem without isolating the shock and then obtain it in an end-to-end calculation with an appropriate method with passing the shock wave structure, that is far from simple at high Reynolds numbers, or reduce the order of the system of Navier–Stokes equations in the variable y by one and then formulate the boundary conditions on the shock. The latter procedure is implemented asymptotically in the two-layer model (the shock layer proper and the shock wave structure) for the problem of flow over a body at both high and moderate Reynolds numbers. It is noteworthy that it is specifically the out-of-order terms containing second derivatives in the normal coordinate y, proportional to Re^{-1}, that drop out in the viscous shock layer model obtained in this case.

As the Reynolds number decreases, the thickness of the Navier–Stokes shock wave structure, which has a value of the order of the Knudsen number, will increase, the classical Rankine–Hugoniot relations, which hold at fairly high Reynolds numbers, i.e., for an ideal gas, should be extended to the case when the viscosity and heat conduction are taken into accoun. These dynamical conditions, which express the laws of conservation of mass, momentum and energy on passing through a shock, are obtained by integrating Eqs (1.1)–(1.4), written in integral form, over the coordinate that is normal to the discontinuity surface[24]:

$$\begin{aligned}
&[\rho\upsilon_n] = 0, \quad [\rho\upsilon_n\mathbf{v} - \mathbf{p}_n] = 0, \quad \mathbf{p}_n = -p\mathbf{n} + \frac{1}{\mathrm{Re}}\hat{\tau}\cdot\mathbf{n} \\
&\left[\rho\upsilon_n H + \frac{1}{\mathrm{Re}}\mathbf{J}\cdot\mathbf{n}\right] = 0, \quad \mathbf{J}\cdot\mathbf{n} = -\frac{\mu}{\sigma}\left[\frac{\partial H}{\partial n} + \frac{\sigma}{\mu}(\hat{\tau}\cdot\mathbf{v})_n - \frac{\partial}{\partial n}\frac{u^2+\upsilon^2}{2}\right]
\end{aligned} \tag{2.8}$$

Here, as usual, the square brackets denote the difference between the values on the two sides of the discontinuity, $[f] = f_\infty - f_s$, $V_\infty \nu_n$ is the projection of the velocity onto a normal to the shock, $V_\infty\mathbf{v}$ is the velocity vector, $\mathbf{n}$ is a unit vector of a normal to the curve $y = y_s(x)$, directed toward the free stream, and p_n is the stress, divided by $\rho_\infty V_\infty^2$, in an area with normal $\mathbf{n}$. If the terms with the viscosity and thermal conductivity in relations (2.8) are retained on the shock side facing the body (the subscript s) and are discarded on the side facing the free stream (the subscript ∞), by virtue of their small values and the low density upstream the shock,[25] we obtain

$$\upsilon_{n\infty} = \rho_s\upsilon_{ns} \tag{2.9}$$

$$\upsilon_{n\infty}\mathbf{V}_\infty + \frac{\mathbf{n}}{\gamma M_\infty^2} = \rho_s\upsilon_{ns}\mathbf{v}_s + p_s\mathbf{n} - \frac{1}{\text{Re}}(\hat{\tau}\cdot\mathbf{n}) \tag{2.10}$$

$$\upsilon_{n\infty}H_\infty = \rho_s\upsilon_{ns}H_s - \frac{\mu_s}{\sigma_s \text{Re}}\left[\frac{\partial H}{\partial n} + \frac{\sigma}{\mu}(\hat{\tau}\cdot\mathbf{v})_n - \frac{\partial}{\partial n}\frac{u^2+\upsilon^2}{2}\right]_s \tag{2.11}$$

Relations (2.9)–(2.11) together with (1.5) give five equations for determining five variables on the shock, viz., u_s, υ_s, p_s, ρ_s and H_s (or T_s), which depend on the angle of inclination of the shock $\beta_s = \beta - \alpha$ and the viscous stress tensor $\hat{\tau}$ immediately behind the shock.

From condition (2.9) and vector condition (2.10), which was taken in a projection onto a tangent to the curve $y = y_s(x)$, we obtain two equations for determining the velocity components u_s and υ_s on the shock ($\upsilon_{n\infty} = -\sin\beta$, $\upsilon_{\tau\infty} = \cos\beta$)

$$\begin{aligned}\upsilon_{ns} &= -k\sin\beta = -u_s\sin\beta_s + \upsilon_s\cos\beta_s\\ \upsilon_{\tau s} &= \cos\beta - \overline{\tau}_{n\tau s} = u_s\cos\beta_s + \upsilon_s\sin\beta_s\end{aligned} \tag{2.12}$$

where

$$k = \frac{\rho_\infty}{\rho_\infty\rho_s} = \frac{1}{\rho_s},\quad \overline{\tau}_{n\tau s} = \frac{\tau_{n\tau s}}{\text{Re}\sin\beta}$$

$$\tau_{n\tau s} = [(\hat{\tau}\cdot\mathbf{n})\cdot\boldsymbol{\tau}]_s = [(\hat{P}\cdot\mathbf{n})\cdot\tau]_s = \left[(\tau_{yy}-\tau_{xx})\text{tg}\beta_s + \tau_{xy}(1-\text{tg}^2\beta_s)\right]_s\cos^2\beta_s$$

Here τ is a unit vector of the tangent to the shock, k is the ratio of the densities in the free stream and behind the discontinuity (on the contour sought $y = y_s(x)$), $\tau_{n\tau s}$ is the projection on a tangent to the shock of the viscous stress vector $\boldsymbol{\tau}_n = \hat{\tau}\cdot\mathbf{n}$ acting on an area with a normal $\mathbf{n}$ to the shock, and $\hat{\mathbf{P}}$ is the stress tensor relative to $\rho_\infty V_\infty^2$.

From equalities (2.12) we find the dimensionless components of the velocity vector on the shock:

$$u_s = u_i - \frac{\cos\beta_s}{\text{Re}\sin\beta}\tau_{n\tau s},\quad \upsilon_s = \upsilon_i - \frac{\sin\beta_s}{\text{Re}\sin\beta}\tau_{n\tau s} = u_s\text{tg}\beta_s - k\frac{\sin\beta}{\cos\beta_s} \tag{2.13}$$

where

$$\begin{aligned}u_i &= \cos\beta\cos\beta_s + k\sin\beta\sin\beta_s = \cos^2\beta_s\left[(1+k\,\text{tg}^2\beta_s)\cos\alpha - (1-k)\text{tg}\beta_s\sin\alpha\right]\\ \upsilon_i &= \cos\beta\sin\beta_s - k\sin\beta\cos\beta = \cos^2\beta_s\left[(1-k)\text{tg}\beta_s\cos\alpha - (k+\text{tg}^2\beta_s)\sin\alpha\right]\\ &= u_i\text{tg}\beta_s - k\sin\beta/\cos\beta_s\end{aligned}$$

Here u_i and υ_i are the components of the velocity vector on the shock along the x and y axes in an ideal gas, i.e., for an infinitely high Reynolds number.

From vector equation (2.10), projected onto a normal to the shock, we obtain

$$p_s = \frac{1}{\gamma M_\infty^2} + (1-k)\sin^2\beta + \frac{1}{\text{Re}}\tau_{nns} \tag{2.14}$$

where

$$\tau_{nns} = [(\hat{\tau}\cdot\mathbf{n})\cdot\mathbf{n}]_s = \left(\tau_{yy} + \tau_{xx}\text{tg}^2\beta_s - 2\tau_{xy}\text{tg}\beta_s\right)_s\cos^2\beta_s$$

Equality (2.11) gives an expression for the total enthalpy on the shock

$$H_s = H_\infty + \frac{1}{\text{Re}\sin\beta}(\mathbf{J}\cdot\mathbf{n})_s \tag{2.15}$$

where

$$\begin{aligned}(\mathbf{J}\cdot\mathbf{n})_s &= (J_y - J_x\text{tg}\beta_s)_s\cos\beta_s\\ J_x &= -\frac{\mu}{\sigma}\left[\frac{1}{H_1}\frac{\partial H}{\partial x} + \frac{\sigma}{\mu}(u\tau_{xx} + \upsilon\tau_{xy}) - \frac{1}{H_1}\frac{\partial}{\partial x}\frac{u^2+\upsilon^2}{2}\right]\\ J_y &= -\frac{\mu}{\sigma}\left[\frac{\partial H}{\partial y} + \frac{\sigma}{\mu}(u\tau_{xy} + \upsilon\tau_{yy}) - \frac{\partial}{\partial y}\frac{u^2+\upsilon^2}{2}\right]\end{aligned} \tag{2.16}$$

Finally, from equation of state (1.5) we obtain

$$\frac{1}{\rho_s} = k = \frac{R_{Al}T}{p_sV_\infty^2} = \frac{\gamma-1}{2\gamma p_s}\frac{T_s}{T_0},\quad T_0 = \frac{V_\infty^2}{2c_p} \tag{2.17}$$

where T_0 is the free-stream adiabatic stagnation temperature (minus the free-stream temperature).

Relations (2.13)–(2.15) transform into the classical Rankine–Hugoniot relations on a shock of zero thickness if the gas viscosity and thermal conductivity are neglected, i.e., if the Reynolds number is allowed to tend to infinity. In this case, for an assigned angle $\beta_s = \beta_s(x)$ the five relations in (2.13)–(2.15) and (2.17) are sufficient to determine the five parameters on the shock. When the viscosity and heat conduction effects, which begin to have an effect at fairly low Reynolds numbers, are taken into account, this cannot be done before solving the problem. In this context we note that the derivatives with respect to a normal to the shock that appear in expressions (2.13)–(2.15) were previously calculated[20] from Euler's equation, and approximate finite expressions for the flow parameters on a shock were thereby obtained. However, it is difficult to assess the accuracy of the results of such an approach *a priori*. Conditions (2.13)–(2.15) are the boundary conditions on a shock wave, which are sometimes called the slip and temperature jump conditions because of the mismatch between the tangential component of the velocity and the total enthalpy H behind the shock and the corresponding quantities in front of the shock due to the influence of the viscosity and heat conduction effects, i.e., $\upsilon_{\tau s} \neq \upsilon_{\tau\infty}$, $H_s \neq H_\infty$. These effects show up at fairly low Reynolds numbers, as can be seen from conditions (2.13)–(2.15). In an ideal gas, i.e., when $\mathrm{Re} \to \infty$, the quantities υ_τ and H remain continuous on passing through the shock. The velocity component normal to the shock, the pressure and the density at $M_\infty > 1$, both in an ideal gas and in a viscous heat-conducting gas, have a discontinuity (jump), but it varies in magnitude due to the viscosity and thermal conductivity, effects.

Boundary condition (2.13) for the velocity υ_s on the shock can be replaced by an equivalent condition, that follows from the mass conservation condition of the gas flowing through the closed contour $ONQQ_s$ (Fig. 1), which can be written in the form

$$\rho_\infty V_\infty \pi^\nu r_s^{\nu+1} = \int_0^{y_s(x)} \rho_\infty \rho \upsilon_x (2\pi r)^\nu dy - \int_0^x (\rho_\infty \rho \upsilon_y)_w (2\pi r_w)^\nu dx$$

or

$$r_s^{\nu+1} = (\nu+1)\int_0^{y_s(x)} \rho u r^\nu dy - (\nu+1)\int_0^x (\rho\upsilon)_w r_w^\nu dx, \quad r_s = r_w + y_s\cos\alpha \tag{2.18}$$

The second term on the right-hand side is proportional to be the total mass rate of injection (or suction) of the gas through the body surface from the vertex of the body to the current value of x. When the Navier–Stokes equations or the viscous shock layer equations are solved with the boundary conditions on an assigned boundary $y = y_s(x)$, condition (2.18) can serve as an additional means for monitoring the accuracy of the solution of the problem.

3. The drag and heat transfer coefficients

After the problem is solved, it is usually necessary to find:the pressure coefficient

$$C_p = \frac{\rho_\infty V_\infty^2 p_w - p_\infty}{\rho_\infty V_\infty^2/2} = 2\left(p_w - \frac{1}{\gamma M_\infty^2}\right) \tag{3.1}$$

the local scin friction coefficient, i.e., the dimensionless friction stress, acting on unit area of the body surface with an outward normal **n**, taking into account the injection (suction) of the gas and the flow slip on the surface ($\upsilon_x(x,0) \neq 0$, $\upsilon_y(x,0) \neq 0$)

$$C_f = \frac{F_\tau}{\rho_\infty V_\infty^2/2} = \frac{2}{\mathrm{Re}_w}(\tau_{xy}\cos\alpha - \tau_{yy}\sin\alpha)_w \tag{3.2}$$

the local and total convective heat transfer coefficients

$$C_H = \frac{q_w}{\rho_\infty V_\infty^3 (H_\infty - H_w)} = \frac{\mu_w}{(H_\infty - H_w)\sigma_w \mathrm{Re}}\left[\frac{\partial H}{\partial y} + \frac{\sigma}{\mu}(u\tau_{xy} + \upsilon\tau_{yy}) - \frac{\partial}{\partial y}\frac{u^2+\upsilon^2}{2}\right]_w$$

$$C_{H\Sigma} = \frac{1}{\rho_\infty V_\infty^3 (H_\infty - H_w)\pi^\nu r_w^{\nu+1}} \int_0^x q_w (2\pi r_w)\nu dx$$

$$= \frac{2^\nu}{(H_\infty - H_w) r_w^{\nu+1}} \int_0^x \frac{1}{\sigma_0 \mathrm{Re}}\left[\frac{\partial H}{\partial y} + \frac{\sigma}{\mu}(u\tau_{xx} + \upsilon\tau_{xy}) - \frac{\partial}{\partial y}\frac{u^2+\upsilon^2}{2}\right]_w r_w^\nu dx \tag{3.3}$$

the local and total drag coefficients

$$C_D = \frac{2F_z}{\rho_\infty V_\infty^2} = \frac{2}{\mathrm{Re}}\left[\tau_{xy}\cos\alpha + \left(p_w - \frac{1}{\gamma M_\infty^2} - \tau_{yy}\right)\sin\alpha\right]$$

$$C_{D\Sigma} = \frac{2F_z}{\rho_\infty V_\infty^2 r_w^{\nu+1}} = \frac{2^{\nu+1}}{\mathrm{Re}} \int_0^x \left[\tau_{xy}\cos\alpha + \left(p_w - \frac{1}{\gamma M_\infty^2} - \tau_{yy}\right)\sin\alpha\right] r_w^\nu dx \tag{3.4}$$

Here F_z is the total force acting on a body with a contour length x in the free-stream direction along the Oz axis (Fig. 1).

4. Generalization of the viscous shock layer equation at moderate and high Reynolds numbers

When $\mathrm{Re}\to\infty$, under the assumption that near the surface the viscosity forces are of the same order as the inertial forces (Prandtl's hypothesis), from Eqs (1.2) and (1.1) we obtain the classical boundary layer estimates: $y \sim \mathrm{Re}^{-1/2}$, $\upsilon \sim \mathrm{Re}^{-1/2}$. Taking these estimates into account and retaining the $O(1)$ and $O(\mathrm{Re}^{-1/2})$ terms with the viscosity and thermal conductivity, i.e., taking the effects of the second approximation of the boundry layer theory into account. and the terms that do not contain second derivatives in the marching coordinate x (the out-of-order terms) in Eqs (1.2)–(1.4), we obtain the following system of generalized viscous shock layer equations

$$\frac{\partial}{\partial x}\left(r^{\nu}\rho u\right)+\frac{\partial}{\partial y}\left(H_1 r^{\nu}\rho\upsilon\right)=0 \tag{4.1}$$

$$\rho\left(Du+\frac{u\upsilon}{RH_1}\right)+\frac{1}{H_1}\frac{\partial p}{\partial x}=\frac{1}{H_1^2 r^{\nu}\mathrm{Re}}\frac{\partial}{\partial y}\left(H_1 r^{\nu}\tau_{xy}\right)+\frac{\nu\sin\alpha}{r\mathrm{Re}}\left(\tau_{xx}-\tau_{\varphi\varphi}\right) \tag{4.2}$$

$$\rho\left(D\upsilon-\frac{u^2}{RH_1}\right)+\frac{\partial p}{\partial y}=\frac{\nu\sin\alpha}{r\mathrm{Re}}\tau_{xy}+\frac{1}{RH_1\mathrm{Re}}\left(\tau_{yy}-\tau_{xx}\right)+\frac{\nu\cos\alpha}{r\mathrm{Re}}\left(\tau_{yy}-\tau_{\varphi\varphi}\right) \tag{4.3}$$

$$\rho DH=\frac{1}{H_1 r^{\nu}\mathrm{Re}}\frac{\partial}{\partial y}\left(H_1 r^{\nu}J_y\right)+\frac{\nu\sin\alpha}{r\mathrm{Re}}J_x \tag{4.4}$$

Here

$$D=\frac{u}{H_1}\frac{\partial}{\partial x}+\upsilon\frac{\partial}{\partial y}$$

The dimensionless components of the viscous stress tensor τ_{ij} $(i, j = x, y, \varphi)$ are given by formulae (1.8) and (1.9), from which we obtain

$$\begin{aligned}
\tau_{xx}-\tau_{\varphi\varphi}&=2\mu\left(e_{xx}-e_{\varphi\varphi}\right)=2\mu\left[\frac{1}{H_1}\frac{\partial u}{\partial x}+\frac{\upsilon}{RH_1}-\frac{\nu}{r}(u\sin\alpha+\upsilon\cos\alpha)\right]\\
\tau_{yy}-\tau_{xx}&=2\mu\left(e_{yy}-e_{xx}\right)=2\mu\left(\frac{\partial\upsilon}{\partial y}-\frac{1}{H_1}\frac{\partial u}{\partial x}-\frac{\upsilon}{RH_1}\right)\\
\tau_{yy}-\tau_{\varphi\varphi}&=2\mu\left(e_{yy}-e_{\varphi\varphi}\right)=2\mu\left[\frac{\partial\upsilon}{\partial y}-\frac{\nu}{r}(u\sin\alpha+\upsilon\cos\alpha)\right]
\end{aligned} \tag{4.5}$$

All the differences in (4.5) are of the order of unity; therefore, the last term on the right-hand side in Eq. (4.2) and the last two terms on the right-hand side in Eq. (4.3) are of the order of Re^{-1}. The last term on the right-hand side of Eq. (4.4) is of the same order. In the asymptotic sense, they will thus be of the third order, and they are omitted in the classical viscous shock layer equations.[7] However, retaining them does not alter the mathematical nature of system of equations (4.1)–(4.4) at either high or moderate Reynolds numbers (Re). When they are taken into account, the solution will be refined and will approximate to the solution of the complete Navier–Stokes equations. We will call system of equations (4.1)–(4.4) the generalized system of viscous shock layer equations.

5. Generalized boundary conditions on a head shock

As can be seen from the overall form of the dynamical compatibility conditions on a head shock (2.13)–(2.15), the viscosity and heat-conduction effects in them are proportional to Re^{-1}. Therefore, when $\mathrm{Re}\to\infty$, these conditions transform into the classical Rankine–Hugoniot conditions for an ideal gas. However, at the moderate and low Reynolds numbers corresponding to altitudes 75 km $\le z \le$ 110 km ($0.1k \le \mathrm{Kn} \le \mathrm{Re} \sim 10^2$, $k=\rho_s^{-1}$) thickening of the shock layer and the boundary layer occurs and their fusion to form a continuous region of non-isentropic flow in the shock layer on the windward side of the body occur.[25] At still higher altitudes, of the order of 120 km ($1 \le \mathrm{Kn} \le 0.175\,\mathrm{M}_\infty$), a transitional flow regime between continuum flow and free-molecule flow is occurs. At altitudes of the order of 130 km ($0.175\,\mathrm{M}_\infty \le \mathrm{Kn} \le 5.24\,\mathrm{M}_\infty$) a nearly free-molecule flow regime occurs.[25]

As recent studies[19] have shown, consideration of the slip velocity and the temperature jump on a body surface and on a head shock greatly extends the range of applicability of the classical viscous shock layer model for calculating heat transfer on the windward side of a cold blunt body moving with hypersonic velocity along the re entry trajectory of the Space Shuttle (with a nose radius ~1 m) up to altitudes of 140–150 km, which correspond to Knudsen numbers Kn_∞ up to 15–20. Therefore, there is basis for asserting that if the $O(\mathrm{Re}^{-1/2})$ terms are taken into account in the generalized Rankine–Hugoniot conditions under the assumption that the derivative $\partial/\partial y = O(\mathrm{Re}^{1/2})$ (as in boundary layer theory), the relations thus obtained and their use as boundary conditions for the generalized viscous shock layer equations (4.1)–(4.4) will refine the solution of these continuum equations in the transitional and nearly free-molecule flow regimes and will possibly extend the range of their application for more rarefied flow regimes.

The exact relations on a shock obtained from relations (2.13)–(2.15) in dimensionless variables will have the form

$$\begin{aligned}
u_s &= u_i - \frac{\cos^3\beta_s}{\mathrm{Re}\sin\beta}\left[(\tau_{yy} - \tau_{xx})\mathrm{tg}\beta_s + \tau_{xy}\left(1 - \mathrm{tg}^2\beta_s\right)\right] \\
\upsilon_s &= u_s \mathrm{tg}\beta_s - k\frac{\sin\beta}{\cos\beta_s} \\
p_s &= \frac{1}{\gamma M_\infty^2} + (1-k)\sin^2\beta + \frac{\cos^2\beta_s}{\mathrm{Re}}\left[\tau_{yy} + \tau_{xx}\mathrm{tg}^2\beta_s - 2\tau_{xy}\mathrm{tg}\beta_s\right] \\
H_s &= H_\infty - \frac{\cos\beta_s}{\mathrm{Re}\sin\beta}\left(J_y - J_x \mathrm{tg}\beta_s\right)
\end{aligned} \tag{5.1}$$

The dimensionless components of the viscous stress tensor and the components of the specific heat flux vector **J** are given by expressions (1.8) and (2.16), respectively. The density ρ_s or the values of $k = 1/\rho_s$ will be calculated from the state equation (2.17).

We will extimate the viscosity and thermal conductivity terms in the first and last two relations in (5.1) under the assumption that

$$\partial u/\partial y = O\left(\mathrm{Re}^{-1/2}\right), \quad \partial H/\partial y = O\left(\mathrm{Re}^{-1/2}\right), \quad \partial \upsilon/\partial y = O(1)$$

(the latter estimate follows from the continuity equation), i.e., preserving the order of the estimate of the viscosity and thermal conductivity terms that follows from Prandtl's hypothesis regarding the equality between the orders of the inertial terms and the viscosity terms, as well as the equality between the orders of the convective terms and the thermal conductivity terms. This will be the upper estimate of the dissipative terms, i.e., the actual value of the remaining viscosity and thermal conductivity terms will not be greater than the value of the principal terms in relations (5.1), which are of the order of unity. Then from relations (5.1) we obtain the generalized conditions on a shock up to terms $O(\mathrm{Re}^{-1})$, in the form

$$\begin{aligned}
u_s &= u_i - \frac{\cos\beta_s \sin 2\beta_s}{\mathrm{Re}\sin\beta}\mu\frac{\partial u}{\partial y} \\
\upsilon_s &= u_s \mathrm{tg}\beta_s - k\frac{\sin\beta}{\cos\beta_s} \\
p_s &= \frac{1}{\gamma \mathrm{M}_\infty^2} + (1-k)\sin^2\beta + \frac{\cos 2\beta_s}{\mathrm{Re}}\mu\frac{\partial u}{\partial y} \\
H_s &= H_\infty - \frac{\mu\cos\beta_s}{\sigma\mathrm{Re}\sin\beta}\left[\frac{\partial H}{\partial y} + (\sigma - 1)\frac{\partial u^2}{\partial y}\right]
\end{aligned} \tag{5.2}$$

Note that in numerical calculations,[5,9,17] the penultimate condition in (5.2) is used without taking into account the viscosity term and the multipliers $\cos\beta_s \sin 2\beta_s$ in the first condition and $\cos\beta_s$ in the last condition are usually replaced by unity. This does not lead to an is appreciable error at fairly high values of Re and/or in the calculations at the immediate vicinity of the stagnation point, where $\beta_s = \beta - \alpha \to 0$.

6. Conclusion

Recently obtained results have shown[19] that simplified Navier–Stokes equations, i.e., viscous shock layer equations with a slip and a temperature jump conditions on a surface and on a head shock, give a correct prediction of the heat fluxes and drag in problems of hypersonic steady laminar flow over blunt bodies in the transitional flow regime up to altitudes of 140–150 km ($\mathrm{Kn}_\infty \sim 15$–$20$) along the re entry trajectory of the Space Shuttle (with nose radius about 1 m).

A new generalized system of viscous shock layer equations has been obtained, retaining the terms $O(1)$ and $O(\mathrm{Re}^{-1/2})$, as well as the terms $O(\mathrm{Re}^{-1})$, which do not increase the order of the system with respect to the spatial coordinates, as $\mathrm{Re} = \rho_\infty V_\infty R_0/\mu_0 \to \infty$. This enables us to use the previously developed[11,12] efficient marching evolution method of global iterations with respect to the longitudinal pressure gradient (two or three iterations) to solve it. Implicit numerical relaxation methods for solving Navier–Stokes equations require 200-300 iterations with respect to a time difference step. The role of the coefficient of bulk viscosity has incidentally been revealed, and we have shown that taking it into account is related to third-order effects.

Generalized Rankine–Hugoniot conditions on a head shock have also been obtained. They refine the approximate classical conditions that are widely used in current aerothermodynamics research and extend the region of applicability of continuum models to the range of the transitional flow regime.

Comparisons of solutions of the viscous shock layer equations show that they are practically identical to the solution of the Navier–Stokes equations. As is clear from the preceding statements, the generalized viscous shock layer equations have the same property. The model obtained from these equations will greatly reduce the time required to calculate flows of multicomponent reacting gases, as does the classical viscous shock layer model.[15]

Solutions of the complete Navier–Stokes equations can be obtained by a method of global iterations with respect to the "elliptical" dissipative terms (the first terms in expressions (1.7)) and by combining them with global iterations with respect to the longitudinal (marching) pressure gradient $\partial p/\partial x$ in the generalized viscous shock layer equations.

Generalized parabolised Navier–Stokes equations are obtained from system of equations (4.1)–(4.4) by adding the term $(1/\mathrm{Re})(\partial/\partial y)\tau_{yy}$ to the right-hand side of Eq (4.3), which increases its order in the normal coordinate by one, and then the free-stream boundary conditions

(2.6) must be stated for this system, as is done for the complete system of Navier–Stokes equations. The head shock will be obtained as a result of an end-to-end calculation from the free-stream conditions to the body surface.

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